Monday
July 9, 2013

Section 6.1
Review of Power Series.

from calculus 2

* convergence
* interval of convergence
* radius of convergence
* absolute convergence

} Ratio test

We will be manipulating series to make them look a certain way.

MacLaurin Series (In text 6.1)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$x \in (-\infty, \infty)$

$$\frac{1}{x-1} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad (-1, 1)$$

geometric series with $r = x$
common ratio $= x$

convergence $|r| < 1$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$(-\infty, \infty)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$(-\infty, \infty)$

Defn of Analyticity

A function is analytic at a point if it can be represented by a power series in $(x-a)$ with either a positive or infinite radius of convergence.

For instance,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \text{ is}$$

analytic at $x=0$

Arithmetic of Power Series
(shifting the summation index)

but doesn't matter
(different k's)

Example
p. 238
#26

$k=n-1$ $\qquad$ $k=n+1$

Rewrite $\sum_{n=1}^{\infty} n c_n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^{n+2}$

as a single sum

issues: index doesn't match
powers are different

$\Rightarrow$ impose power of x is

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2}$$

$k = n-1$

$n = k+1$

and if $n=1$, then $k=0$

$k = n+2$

$k-2 = n$

If $n=0$, $k=2$

so for

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^{k} + 3\sum_{k=2}^{\infty} c_{k-2} x^{k}$$

start expanding this one

note:

Check

$$\sum_{n=1}^{\infty} n c_k x^{n-1} = \underset{(n=1)}{c_1} + \underset{(n=2)}{2c_2 x} + \underset{(n=3)}{3c_3 x^2} + \cdots$$

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^{k} = \underset{(k=0)}{c_1} + \underset{(k=1)}{2c_2 x} + \underset{(k=2)}{3c_3 x^2} +$$

Recall $\displaystyle\sum_{i=0}^{\infty} a_i = a_0 + a_1 + \sum_{i=2}^{\infty} a_i$

so $\displaystyle\sum_{k=0}^{\infty} (k+1) c_{k+1} x^{k}$

$$= \underset{(k=0)}{c_1} + \underset{(k=1)}{2c_2 x} + \sum_{k=2}^{\infty} (k+1) c_{k+1} x^{k} +$$

$$+ \sum_{k=2}^{\infty} 3 c_{k-2} x^k$$

bring 3 inside → Distributive Prop. of mult over additon

$$3(a_0 + a_1 + a_2 \cdots) = 3a_0 + 3a_1 + \cdots$$

$$= c_1 + c_2 x + \sum_{k=2}^{\infty} \left((k+1) c_{k+1} + 3 c_{k-2} \right) x^k$$

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238 #30

If $n=2, k=2$
$k=n$
$n=k$

If $n=2, k=0$
$k=n-2$
$k+2=n$

① $\sum_{n=2}^{\infty} n(n-1) c_n x^n$ + ② $2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

③ $+ 3 \sum_{n=1}^{\infty} n c_n x^n$

$R=$

$k=n$
$n=k$
If $n=1$, then $k=1$

part 1

$k=n$
$n=k$
If $n=2$, then $k=2$

$$\sum_{k=2}^{\infty} k(k-1) c_k x^n$$

goal x^k

part 2

$$+2\sum_{k=0}^{\infty}(k+2)(k+1)\,c_{k+2}\,x^k$$ 2 terms

part 3

$$+3\sum_{k=1}^{\infty} k c_k x^k$$ 1 term

now set summation start to $k=2$ (biggest)

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \cancel{2(2)(1)\,c_2 x^0}\ \ 4c_2$$

$$+ \cancel{2(3)(2)\,c_3 x^1}\ \ 12c_3 x + \sum_{k=2}^{\infty} 2(k+2)(k+1)\,c_{k+2}\,x^k$$

$$+ \cancel{3(1)\,c_1 x^1}\ \ 3c_1 x + \sum_{k=2}^{\infty} 3k\,c_k\,x^k$$

$$= 4c_2 + 12c_3 + 3c_1 x +$$

$$+ \sum_{k=2}^{\infty}\left[k(k-1)c_k + 2(k+2)(k+1)\,c_{k+2} + 3kc_2\right]x^k$$

$\boxed{\text{Section 6.2}}$

How do we we
apply this to DE?

$\boxed{\text{Solve } y' + y = 0 \text{ using power series}}$

What is a power series?

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x^1 + \cdots$$

looks a lot like a polynomial

so $$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

switch the starting value of the index to $n=1$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1}$$

$$\boxed{y' + y = 0} \text{ implies}$$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n} = 0$$

$k = n-1$ $\qquad$ $k = n$

$n = k+1$ $\qquad$ $n = k$

$n=0, \; k=0$

$$\sum_{n=1}^{\infty} \qquad \sum_{k=0}^{\infty} c_k x^k$$

so

$k = 1-1 = 0$

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} \big(\underbrace{(k+1) c_{k+1} + c_k}_{0}\big) x^k = 0$$

never zero

no adjustment needed $\Longrightarrow$ yields Recursive Formula

so $(k+1) c_{k+1} + c_k = 0$

so $\boxed{c_{k+1} = \dfrac{-c_k}{k+1}}$

Solve for one c

If $k=0$ $\quad c_1 = \dfrac{-c_0}{0+1} = c_1 = -c_0$

$k=0 \quad C_1 = \dfrac{-C_0}{0+1} = -C_0$

$k=1 \quad C_2 = \dfrac{-C_1}{1+1} = -\dfrac{1}{2}C_1 = +\dfrac{1}{2}C_0$

$k=2 \quad C_3 = \dfrac{-C_2}{2+1} = \dfrac{-1}{3}C_2 = \dfrac{-1}{3}\left(\dfrac{1}{2}C_0\right) = \dfrac{-1}{6}C_0$

$k=3 \quad C_4 = \dfrac{-C_3}{3+1} = \dfrac{-1}{4}C_3 = -\dfrac{1}{4}\left(\dfrac{-1}{6}\right)C_0$

$= \dfrac{1}{24}C_0$

$k=4 \quad C_5 = \dfrac{-C_4}{4+1} = \dfrac{-1}{5}C_4 = \dfrac{-1}{120}C_0$

So the solution is:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$= c_0 - c_0 x + \frac{1}{2}c_0 x^2 + \left(-\frac{1}{6}c_0\right)x^3 + \left(\frac{1}{24}c_0\right)x^4 + \left(\frac{-1}{120}c_0\right)x^5$$

$$= c_0\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 \cdots\right)$$

use 3 to 4
terms at least

now rewrite with factorials

$$= c_0\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)$$

from MacClauren series $\rightarrow e^x$

but off by ~~sign~~ sign (calc 1)

$$= c_0 e^{-x}$$

$$y' + y = 0$$

Aux Eqn $\quad m + 1 = 0$

$$m = -1$$

so $y = c_1 e^{-x}$

Auxillary coefficients are usually the easiest.

#7
p244

$y'' - (1+x)y = 0$

no nice auxillary eqn.
(tools are limited)

find sum to represent

always starts the same way

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - (1+x) \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n-2$

$k+2 = n$

$n=2,\ k=0$

$$\sum_{k=0}^{\infty} (k+2)(k+2-1) c_{k+2} x^{k}$$

$k+1$

$$(1+x)\sum_{n=0}^{\infty} c_n x^n$$

expand x into first

$k=n$

$n=k$

$n=0,\ k=0$

$$(1+x)\sum_{k=0}^{\infty} c_k x^k$$

so

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - (1+x)\sum_{n=0}^{\infty} c_k x^k$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{n=0}^{\infty} c_k x^k - x\sum_{n=0}^{\infty} c_k x^k$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty} c_k x^k - x\sum_{k=0}^{\infty} c_k x^k$$

$$\cancel{\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} - c_k(1+x)x^k}$$

need to be k

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^{k+1}$$

$$\sum_{k=1}^{\infty} c_{k-1} x^k$$

1st

remove term

add 1 to all indexes.

Recurrance Formula

$$\text{So } c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}$$

Finish on wednesday.

Wednesday
July 3, 2013

$y'' - (1+x)y = 0$

$y'' - y - xy = 0$ ⟵ distribute first

then document

$y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad y'' = \sum_{n=1}^{\infty} n(n-1) x^{n-2}$

can deal with sums all starting at zero
or getting all to have $\underline{\underline{x^k}}$

$\sum_{k=0}^{\infty}$

set to 0

$\underbrace{2(1)c_2x^0 - c_0x^0}_{0} + \sum_{k=1}^{\infty} \underbrace{[(k+2)(k+1)c_{k+2} - c_k - c_{k-1}] x^k}_{=0}$

$c_2 = \frac{c_0}{2} \qquad c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}$

good tradition is to solve for the c
with the highest index
(ok to solve for others)

$\boxed{\text{goal for Solution}}$

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

If $k=1$ $\quad c_3 = \dfrac{c_0 + c_1}{(3)(2)}$

$k=2$ $\quad c_4 = \dfrac{c_1 + c_2}{(4)(3)}$

looking for patterns

$k=3$ $\quad c_5 = \dfrac{c_2 + c_3}{(5)(4)}$

can get it down to c_0 & c_1 but still a problem

must fit

$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

So arbitrarily assign a value to c_0 & c_1

recall $\boxed{c_2 = \dfrac{c_0}{2}}$

So let $c_1 = 0$

$\boxed{c_3} = \dfrac{c_0}{6}$

$\boxed{c_4} = \dfrac{c_2}{12} = \dfrac{c_0}{24}$

$\boxed{c_5} = \dfrac{c_3 + c_2}{(5)(4)} = \dfrac{\frac{c_0}{6} + \frac{c_0}{2}}{20}$

$= \dfrac{8c_0}{(12)(20)} = \dfrac{4 \cdot 2 c_0}{2 \cdot 4 \cdot 3 \cdot 5 \cdot 2} = \dfrac{c_0}{30}$

now

let $C_0 = 0$

one of these only a on test

so $C_2 = \frac{C_0}{2} = \frac{0}{2} = 0$

$K=1 \quad C_3 = \frac{C_1 + C_0}{6} = \frac{C_1}{6}$

$K=2 \quad C_4 = \frac{C_2 + C_1}{12} = \frac{C_1}{12}$

$K=3 \quad C_5 = \frac{C_3 + C_2}{20} = \frac{\frac{C_1}{6} + 0}{20} = \frac{C_1}{120}$

~~oneset~~ in terms of C_0, other in terms of C_1

$C_0 y_1 = C_0 + \frac{C_0}{2}x^2 + \frac{C_0}{6}x^3 + \frac{C_0}{24}x^4 + \frac{C_0}{15}x^5 + \dots$

$= C_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{15}x^5 + \dots\right)$

$C_1 y_2 = C_1 x + \frac{C_1 x^3}{6} + \frac{C_1}{12}x^4 + \frac{C_1}{60}x^5 + \dots$

$= C_1\left(x + \frac{x^3}{6} + \frac{1}{12}x^4 + \frac{1}{60}x^5 + \dots\right)$

$$\boxed{\text{so } y = C_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) + C_1\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5 + \dots\right)}$$

you only get good at this by practicing! (correct coefficients) first 3 or 4 terms

1) develop series

2) get indicies to match

3) recursion formula

4) some condition ($C_0=0$)

5) general form

$$\sum_{K=1}^{\infty} \qquad \sum_{K=3}^{\infty}$$

start with K=1, etc

[start with K=3]

$\boxed{\text{Quiz at end of class (due next week)}}$